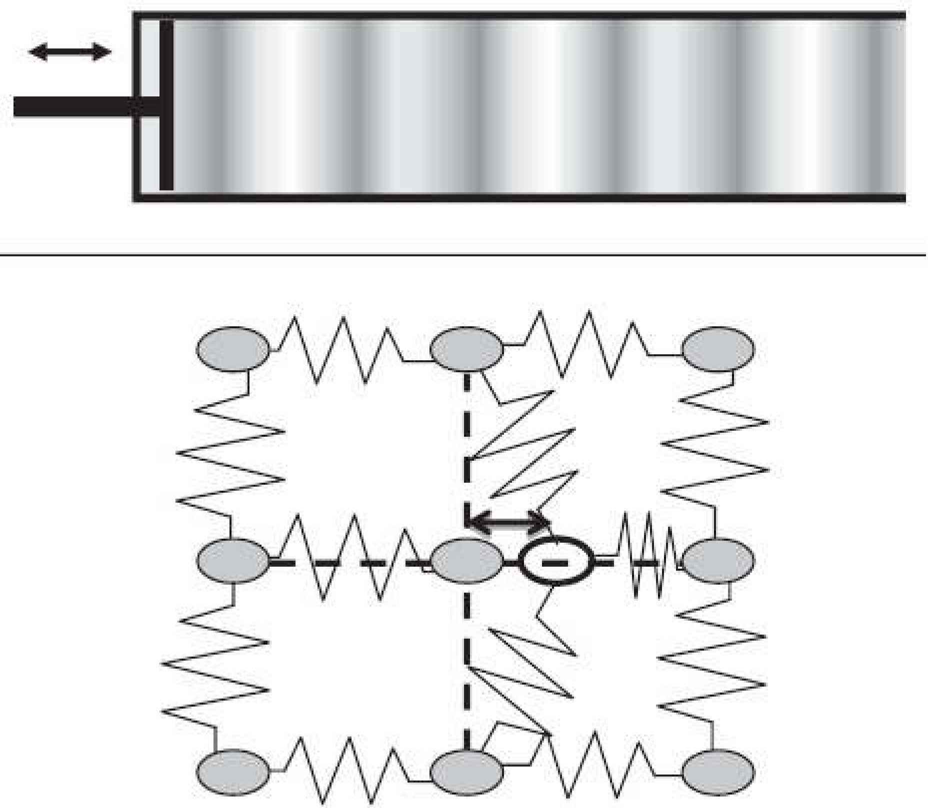
Acoustic wave propagation

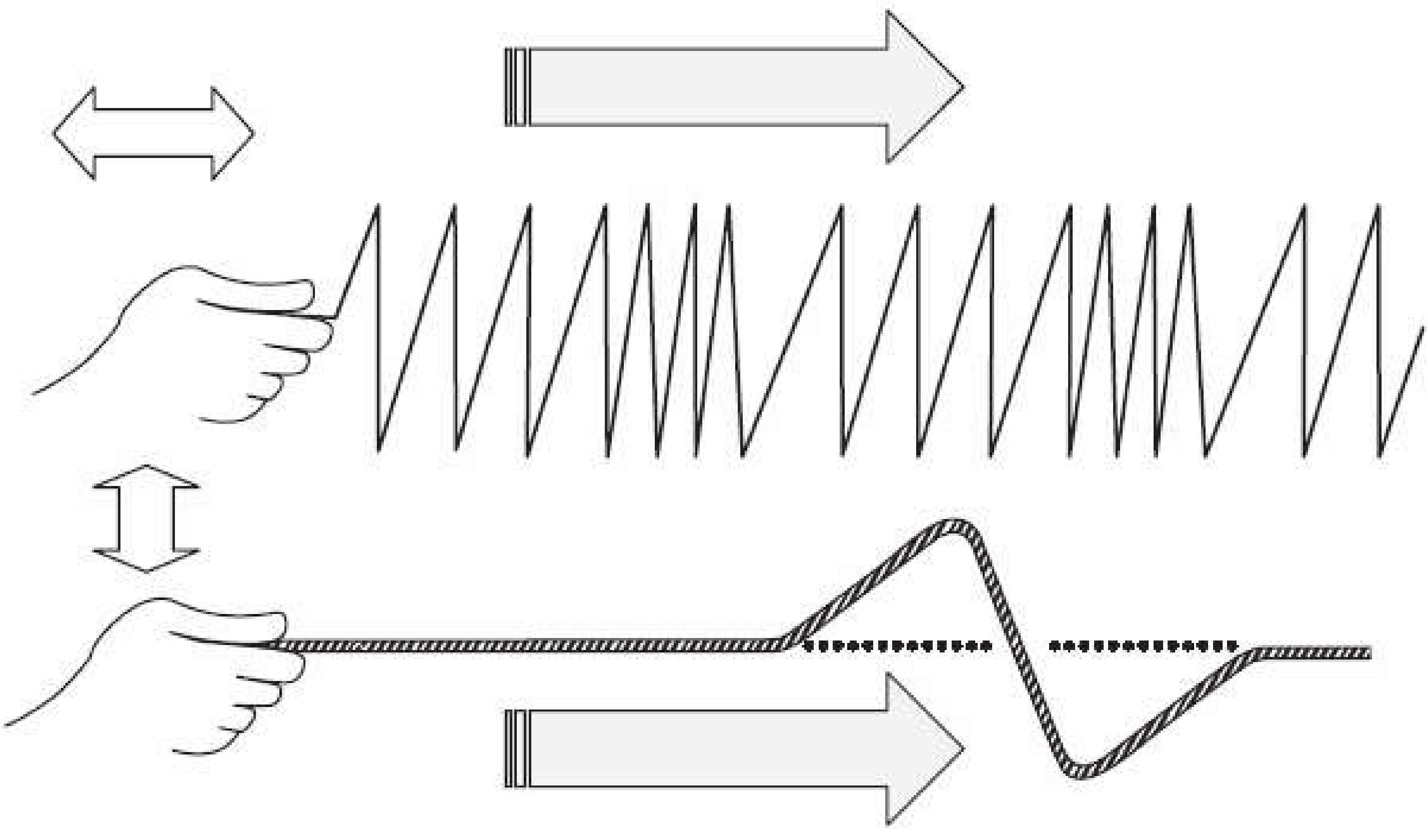
Zahra Kavehvash

Possible configurations of mechanical wave propagation

Possible propagation configurations

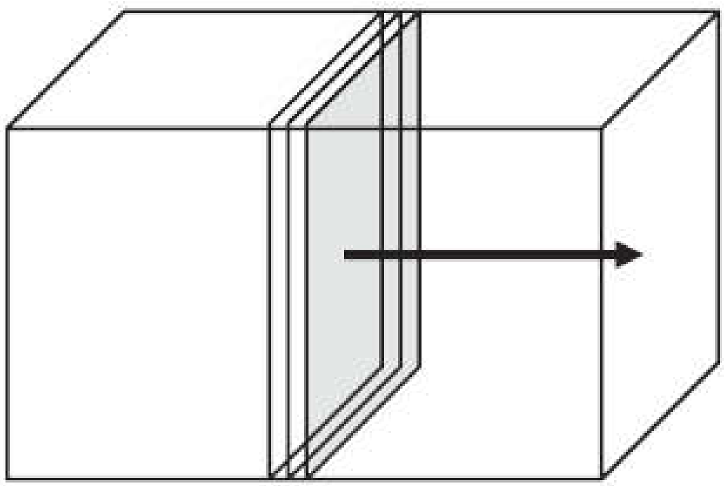


Longtudinal vs. Transverse waves

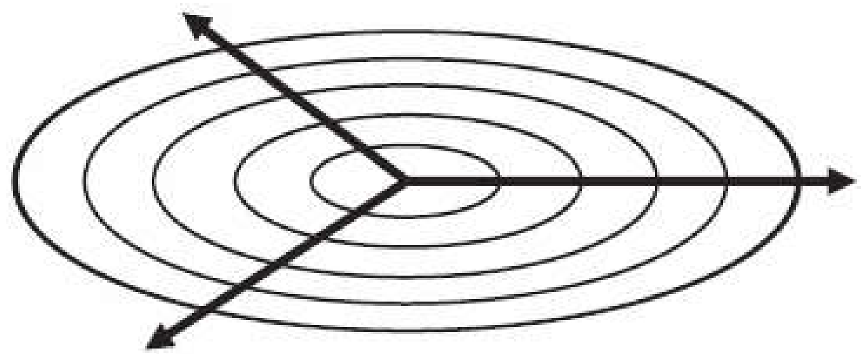


Planar vs. Circular waves

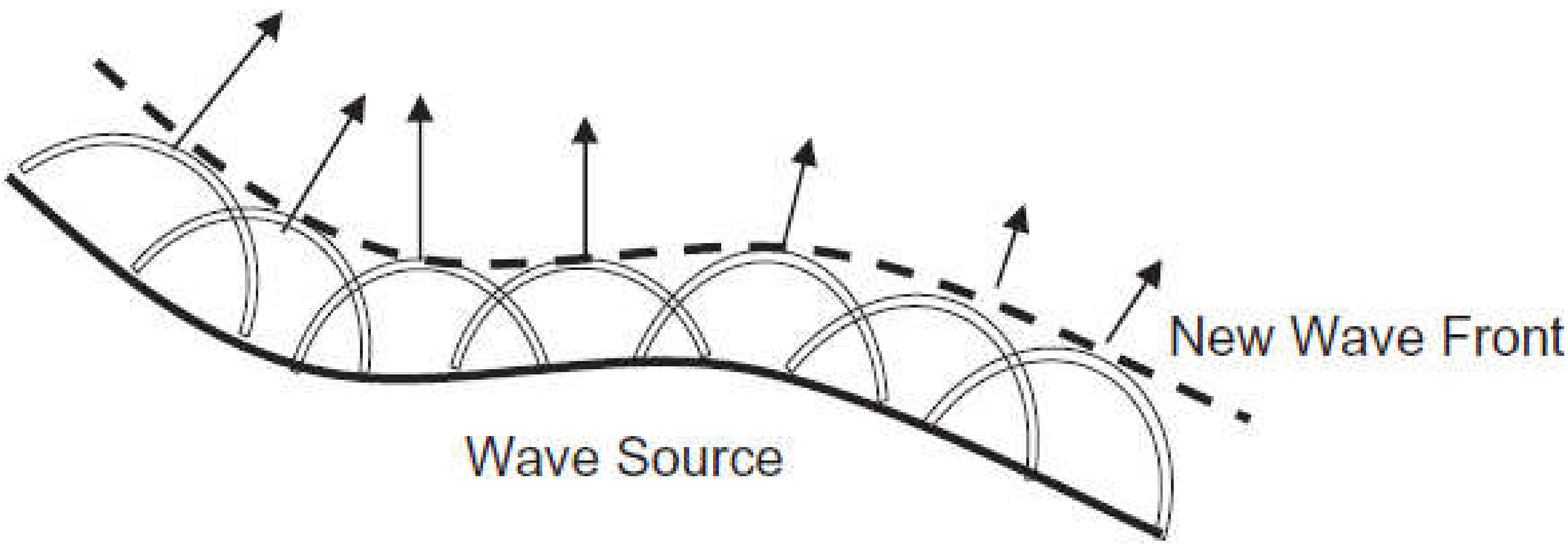
Planar Wave



Circular Wave

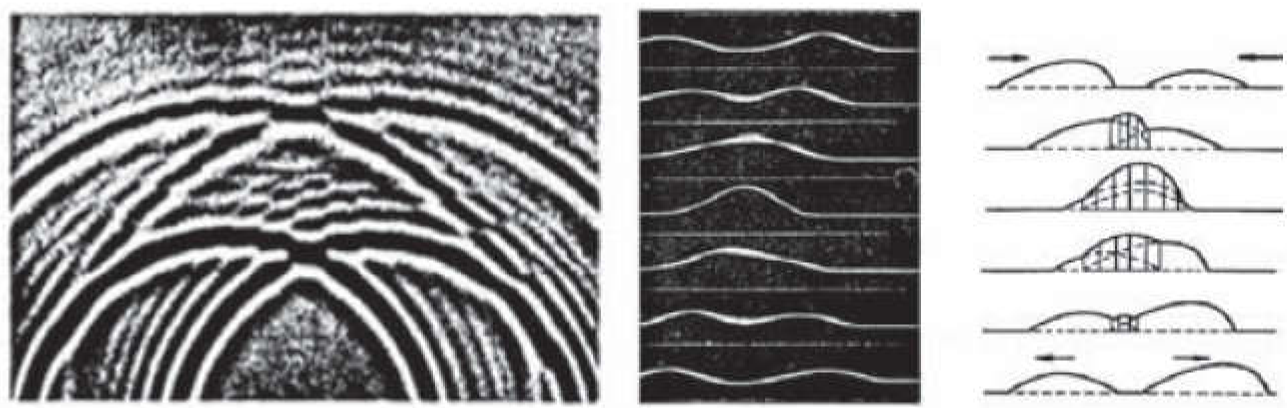


Huygens' Principle



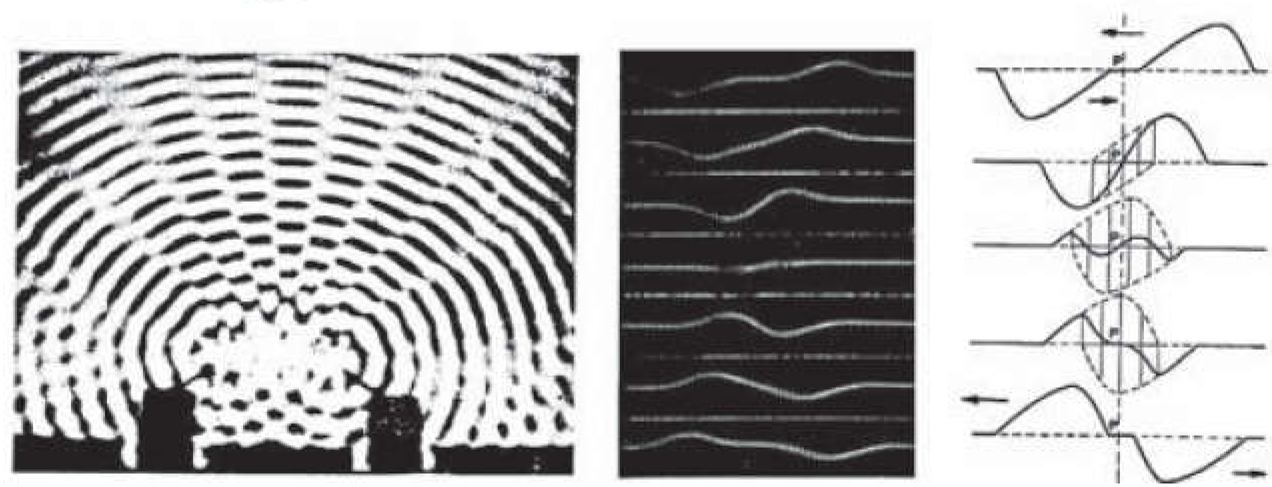
Constructive and Destructive interferences

"constructive interference"

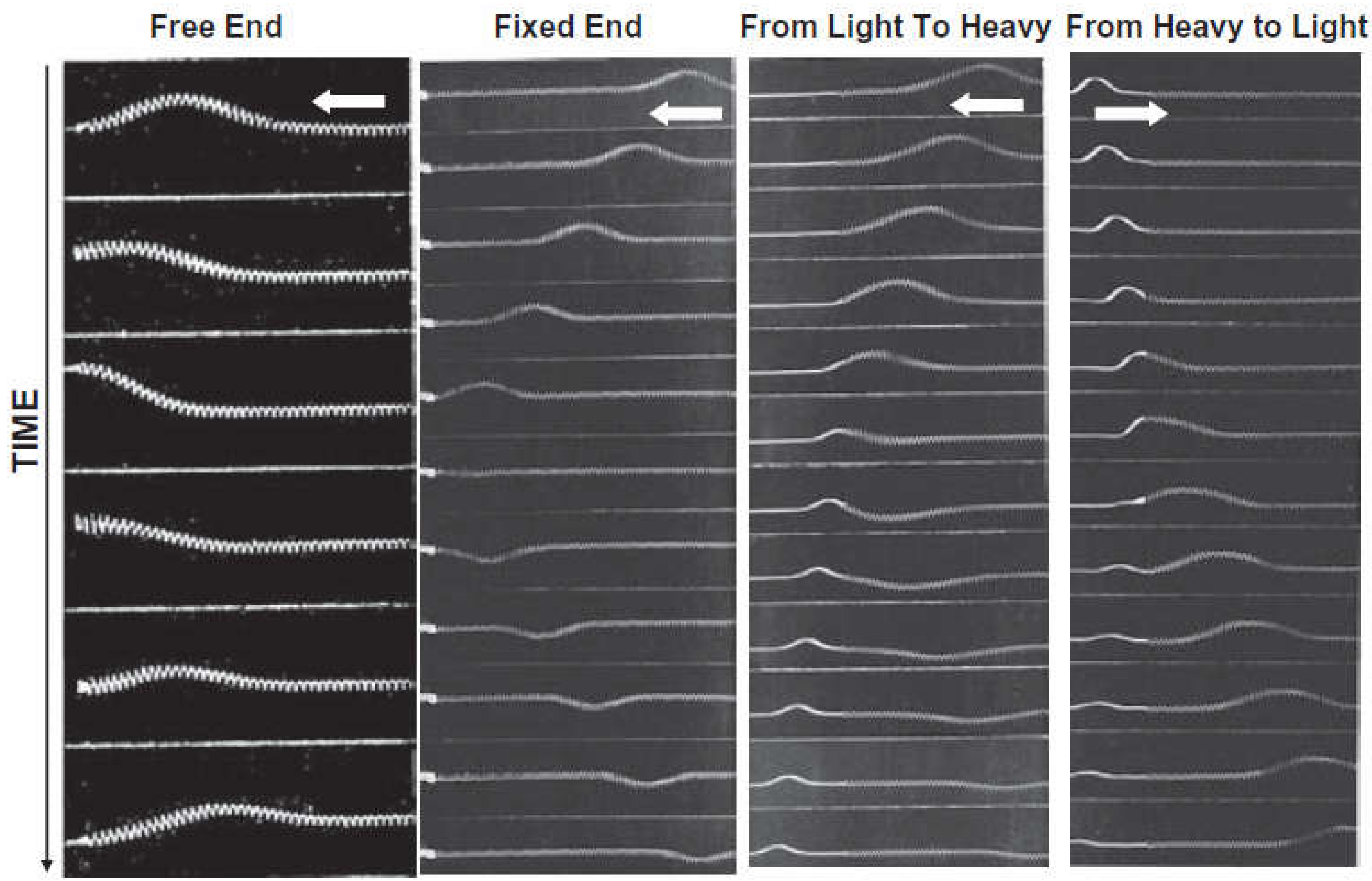


2D 1D

"destructive interference"

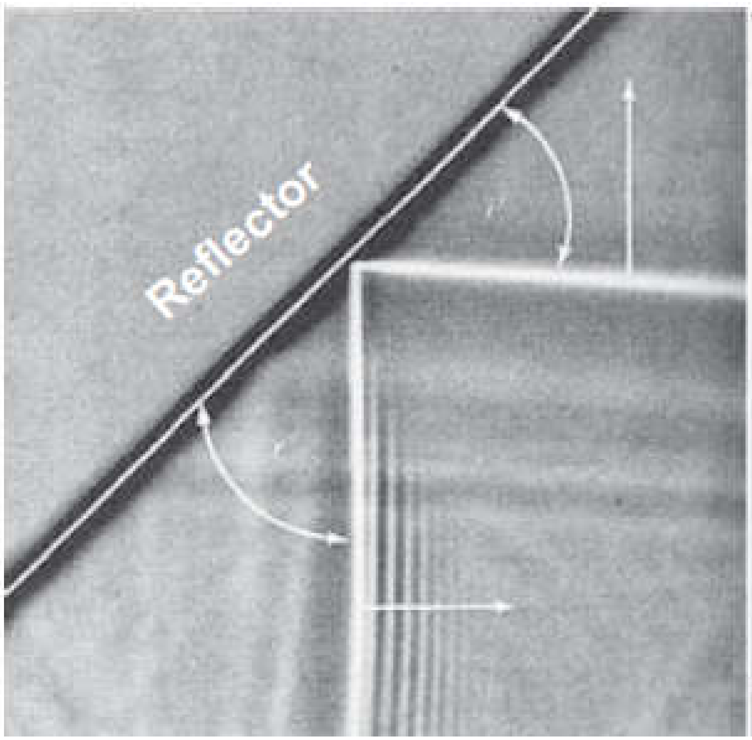


Reflection and Transmission of waves

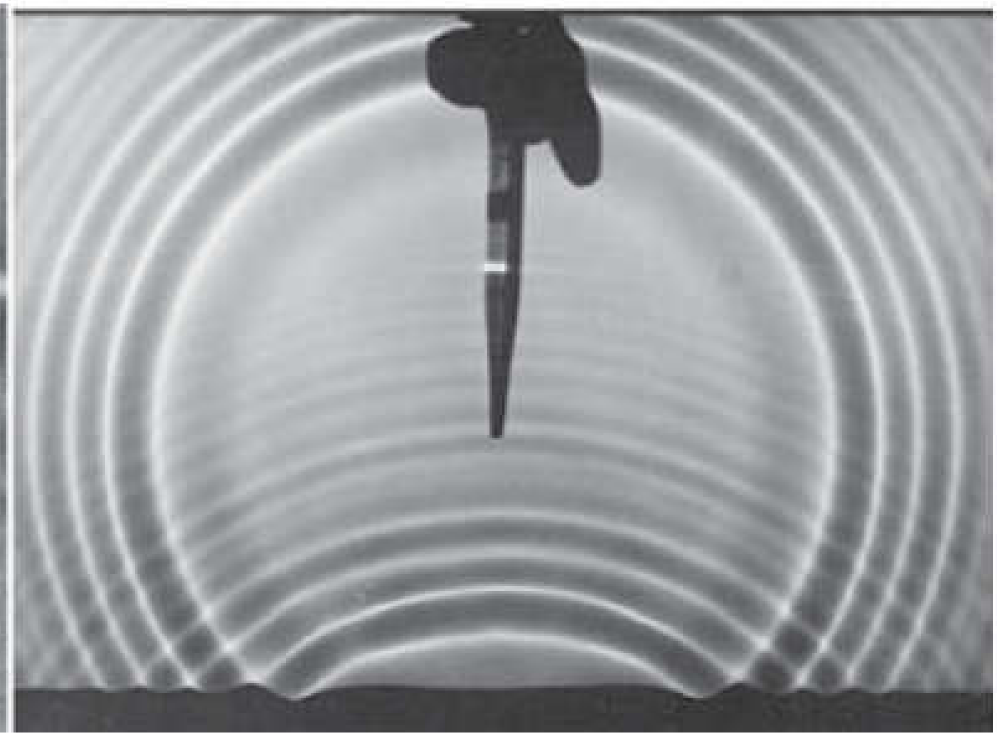


Reflection of planar and spherical waves

Planar Wave Reflection

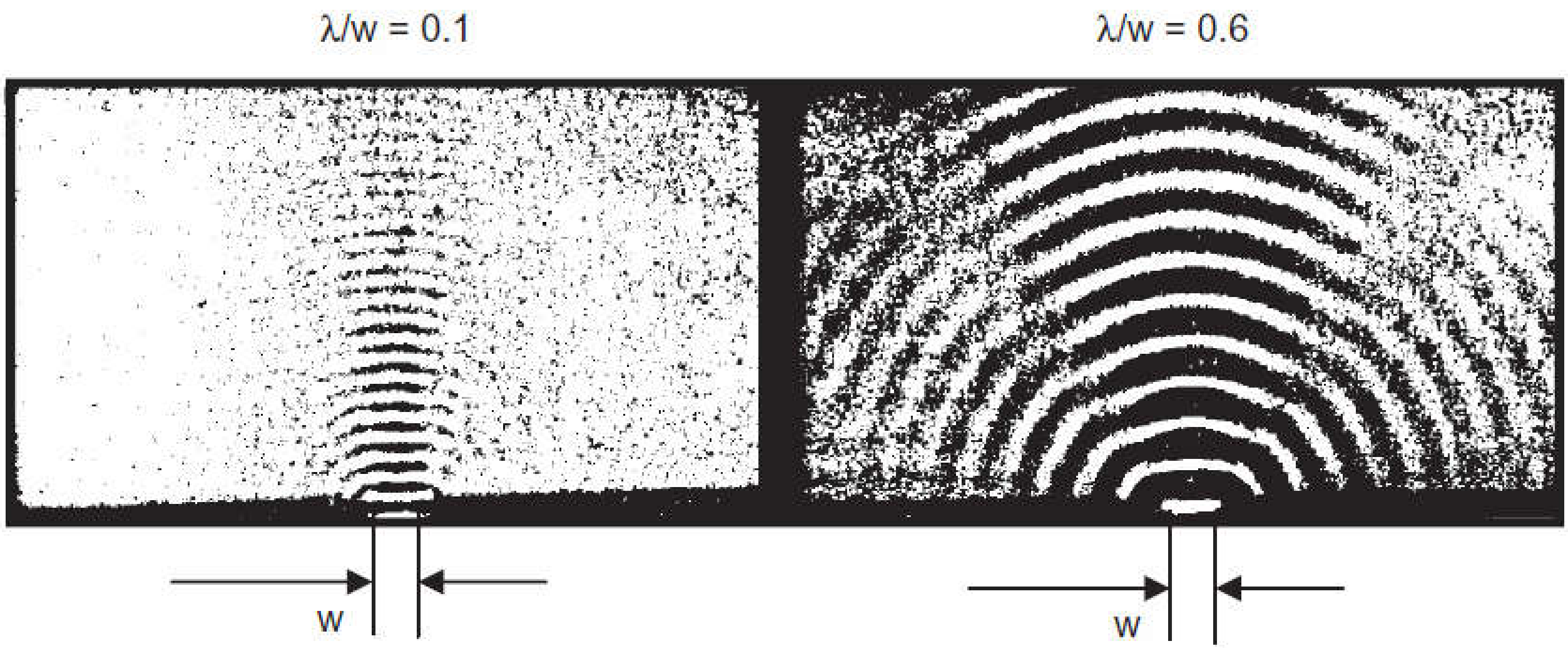


Spherical Wave Reflection

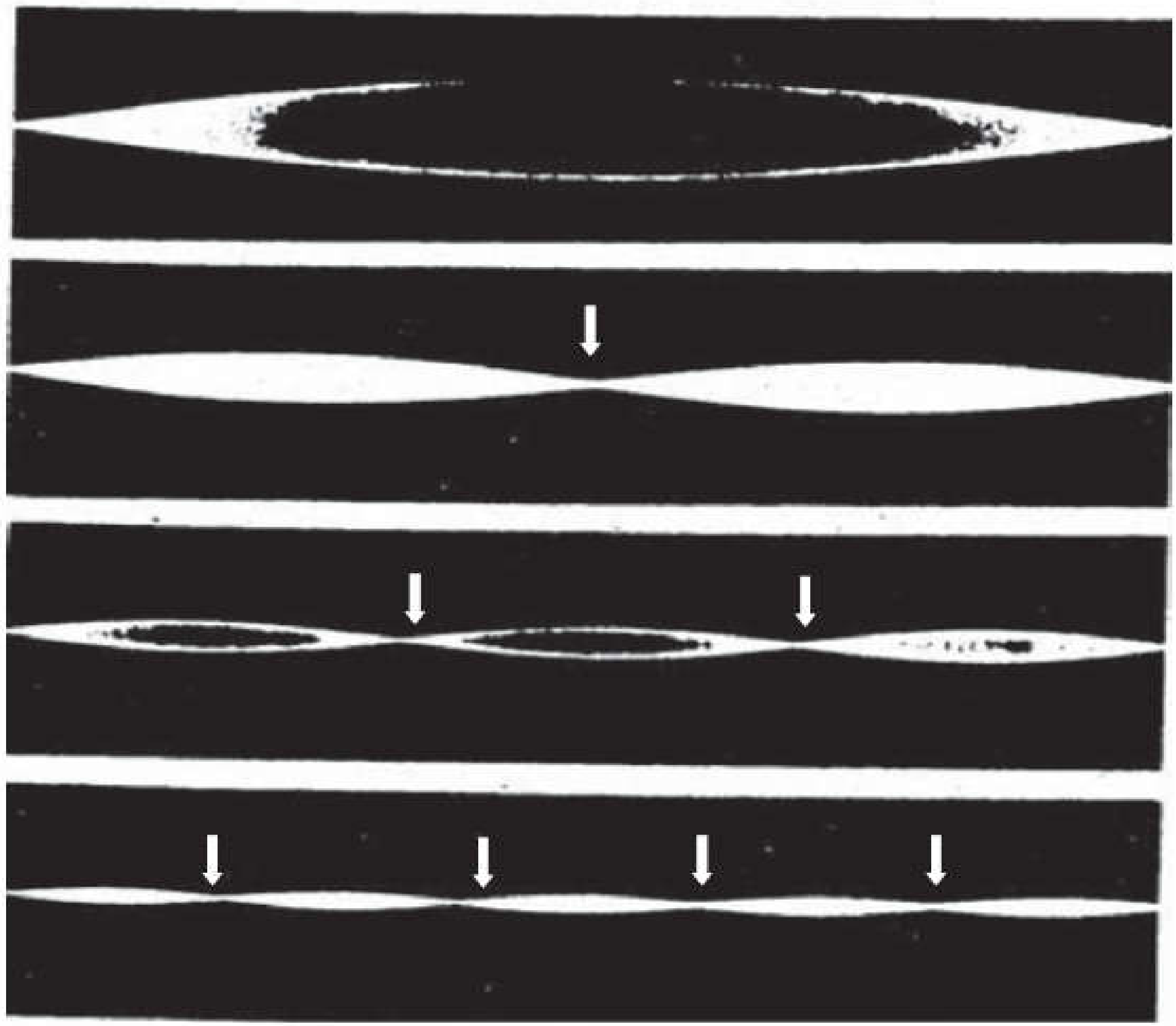


Reflector

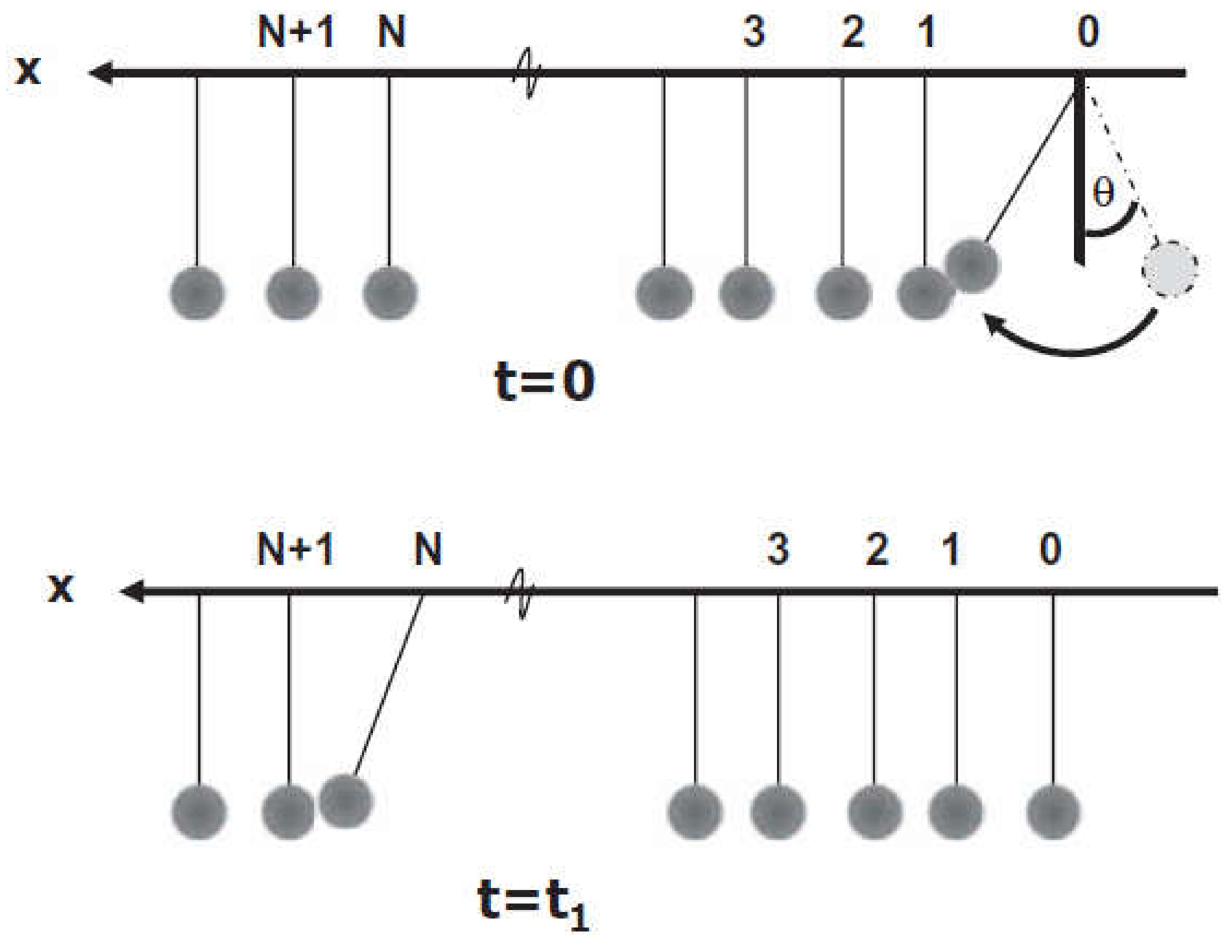
Diffraction



Standing Waves



Mechanical 1D waves



Mechanical 1D waves

- Stems from a perturbation induced to the medium
- Refer to as a wave: $U = U(x, y, z, t)$
  - Pendulum angle relative to its equilibrium state
  - The metal ball velocity
  - The ball energy
- In 1D case: $U = U(x, t)$

The wave function

- Propagation along positive x direction:

$$U(x,t) = U(x - ct)$$

- Propagation along negative x direction:

$$U(x,t) = U(x + ct)$$

The wave equation

$$\frac{\partial^2 U}{\partial X^2} = \frac{1}{C^2} \cdot \frac{\partial^2 U}{\partial t^2} \qquad \text{(one dimension)}$$

- Define: $x - ct \equiv g$ and $\frac{\partial u}{\partial g} = U'$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial g} \cdot \frac{\partial g}{\partial x}$$

- $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial g} \cdot \frac{\partial g}{\partial t}$, since $\frac{\partial u}{\partial t} = -C \cdot \frac{\partial u}{\partial g}$ thus:

$$U'' = \frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$$

The wave equation

- For 3D case:

$$U = U(\hat{n} \cdot \bar{r} - ct)$$

$$\nabla^2 U = \frac{1}{C^2} \cdot \frac{\partial^2 U}{\partial t^2} \qquad \text{(three dimensions)}$$

Harmonic waves

$$u = Ae^{j[\omega t - kx]} = A[\cos(\omega t - kx)) + j\sin(\omega t - kx))]$$

- More generally for 3D:

$$u = Ae^{j[\omega \cdot t - \bar{k} \cdot \bar{R}\ ]}$$

$$\omega = 2\pi f = \frac{2\pi}{T}$$

$$\bar{k} \triangleq k \cdot \hat{n}$$

Harmonic waves

- Group waves:

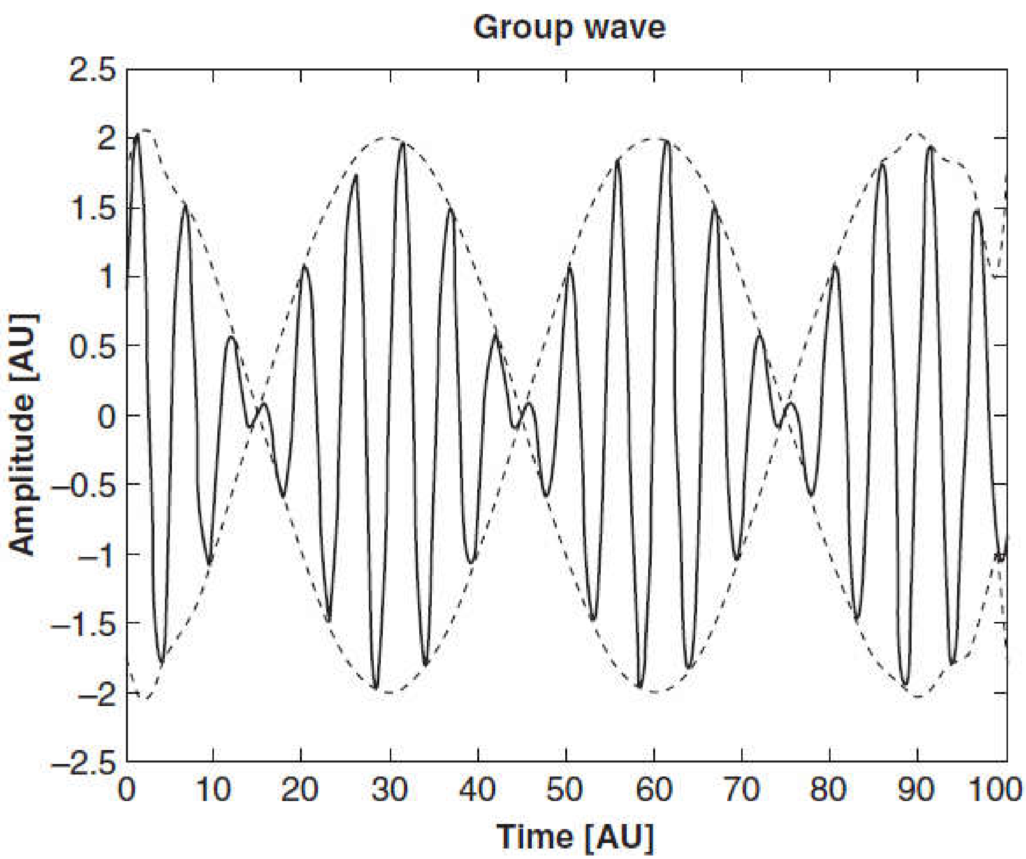
$$U_1 = A\cos(k_1 x - \omega_1 t)$$

$$U_2 = A\cos(k_2 x - \omega_2 t)$$

$$U = 2A\left\{\underbrace{\cos\left[\left(\frac{k_1+k_2}{2}\right)x-\left(\frac{\omega_1+\omega_2}{2}\right)t\right]}_{\text{High-frequency components}}\cdot\underbrace{\cos\left[\left(\frac{k_1-k_2}{2}\right)x-\left(\frac{\omega_1-\omega_2}{2}\right)t\right]}_{\text{Low-frequency components}}\right\}$$

Harmonic waves

- Group waves:



Harmonic waves

- Wave velocity:

$$c = \frac{\Delta x}{\Delta t}$$

- Group velocity: $c_g = \frac{\partial \omega(k)}{\partial k}$
- Phase velocity: $c_p = \frac{\omega(k)}{k}$
- If the speed of sound varies with the wave frequency, these are not the same.

Harmonic waves

- Standing waves:

$$U_1 = A\cos(kx - \omega t)$$

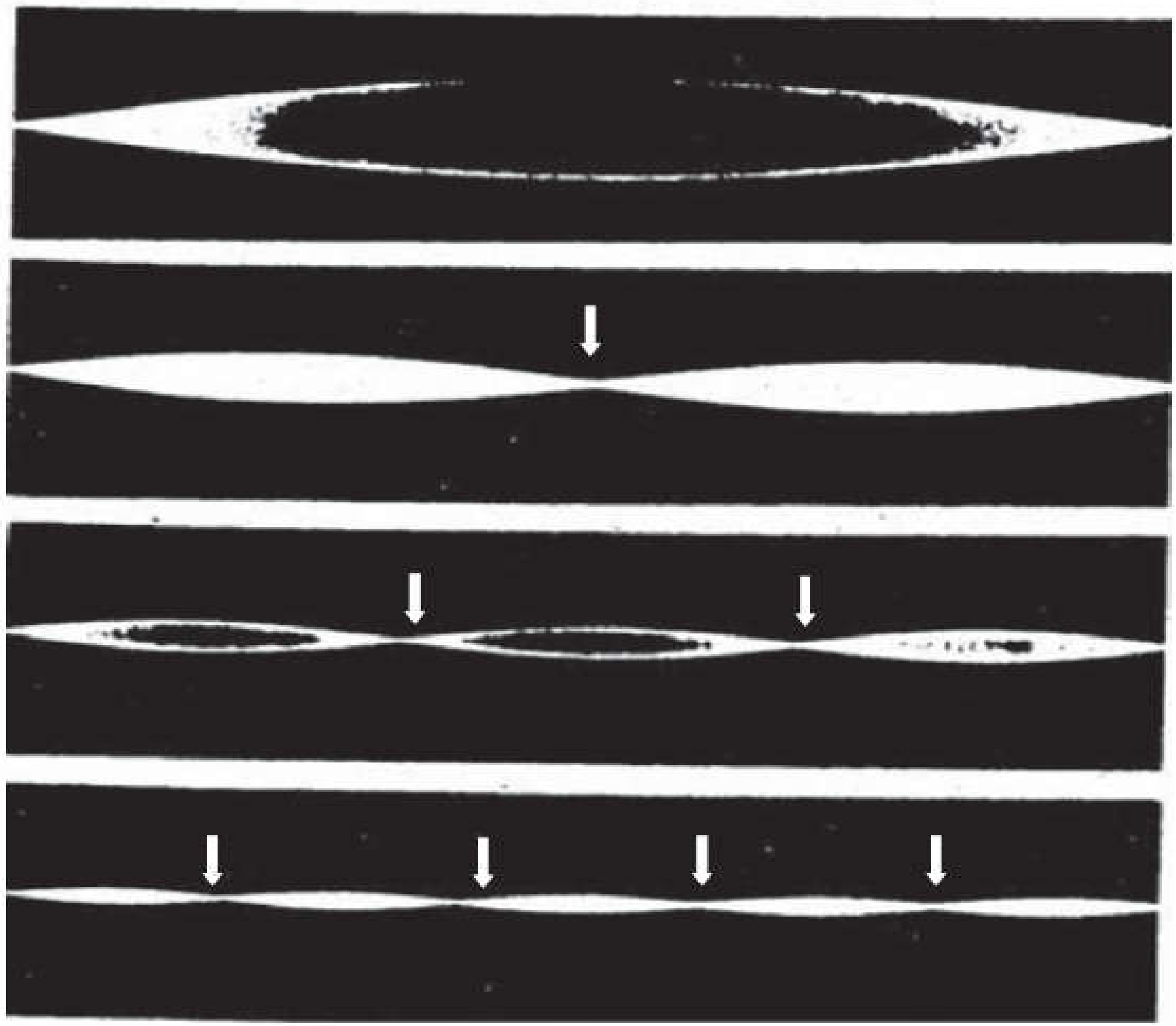
$$U_2 = A\cos(kx + \omega t)$$

$$U = U_1 + U_2$$

$$U = U_1 + U_2 = \underbrace{2A\cos(kx)}_{X(x)} \cdot \underbrace{\cos(\omega t)}_{\Theta(t)}$$

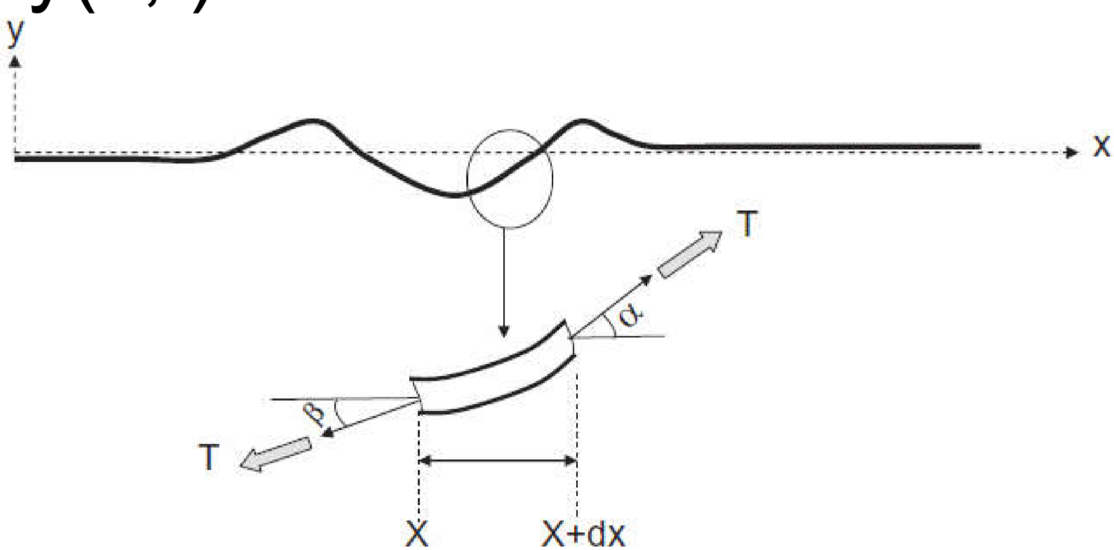
- More generally: $U = X(x) \cdot \Theta(t)$

Standing Waves



Wave in a 1D medium

- Transverse waves in a string:
- Tension (T/Newton)
- Density (ρ - kg/m3)
- Wave function: y(x,t)



Wave in a 1D medium

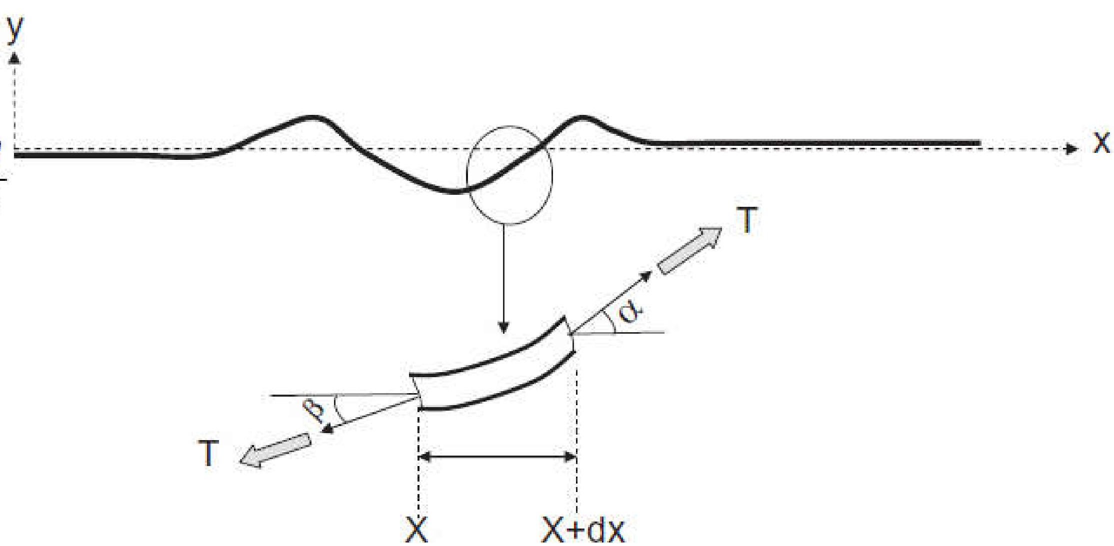
- Newtons' second law: $\sum F_y = m\dfrac{\partial^2 y}{\partial t^2}$

$$-T\sin(\beta)+T\sin(\alpha)=(\rho\,dx)\frac{\partial^2 y}{\partial t^2}$$

$$\tan\alpha=\left.\frac{\partial y}{\partial x}\right|_{x=x+dx}$$

$$\tan\beta=\left.\frac{\partial y}{\partial x}\right|_{x}$$

$$-T\left.\frac{\partial y}{\partial x}\right|_{x}+T\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x}=\rho\,dx\frac{\partial^2 y}{\partial t^2}$$



Wave in a 1D medium

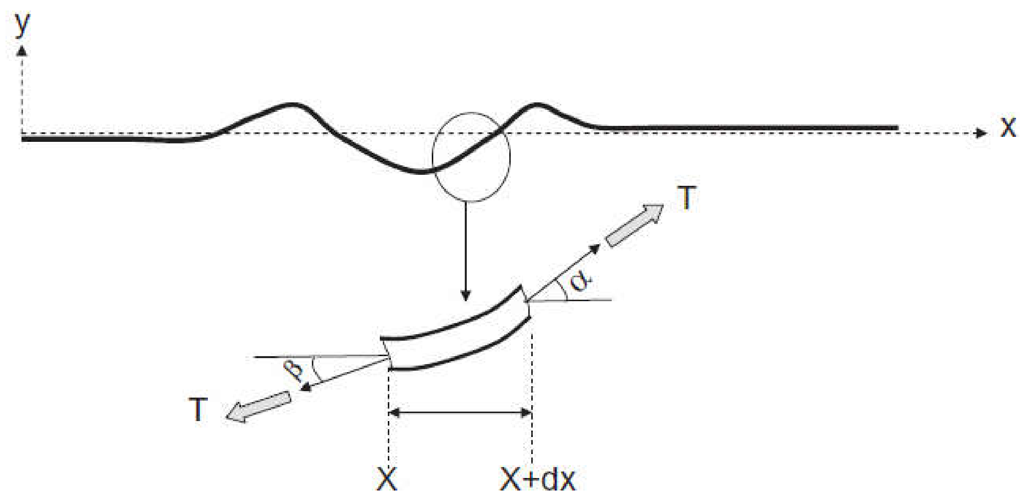
$$\underbrace{\frac{\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - \left.\frac{\partial y}{\partial x}\right|_{x}}{dx}}_{\frac{\partial^2 y}{\partial x^2}} = \frac{\rho}{T} \cdot \frac{\partial^2 y}{\partial t^2}$$

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \cdot \frac{\partial^2 y}{\partial t^2}}$$

- The familiar wave Equation, thus:

$$1/C^2 = \rho/T$$

$$\boxed{c = \sqrt{\frac{T}{\rho}}} \; /T$$

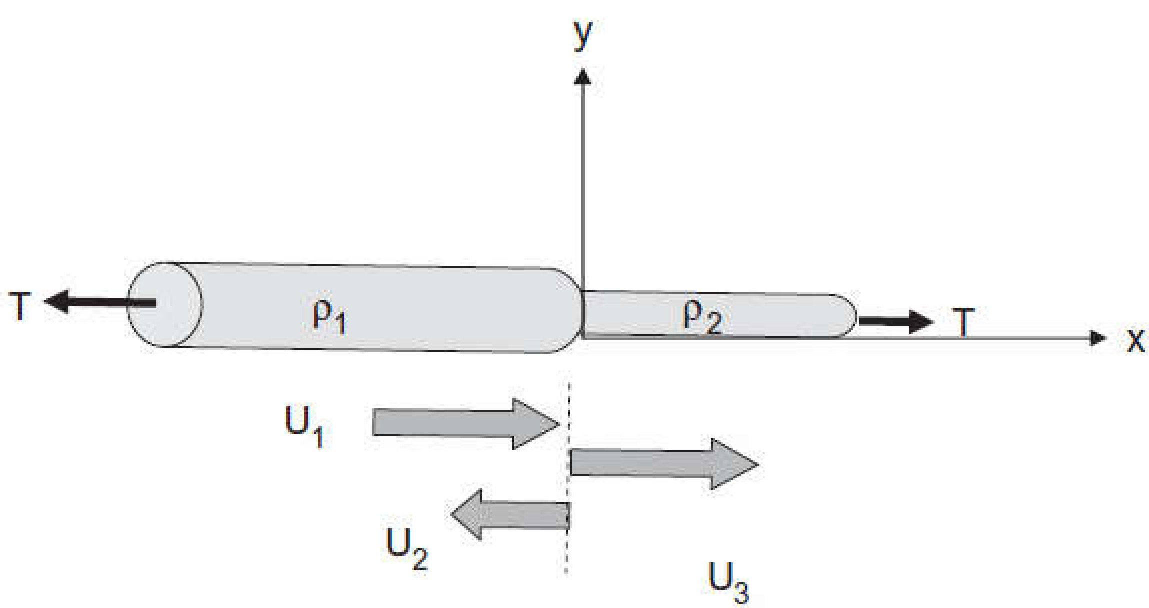


Wave reflection (echo) in a 1D medium

$$U_1 = A_1 \cdot e^{j\omega\left(t - \frac{x}{c_1}\right)}$$

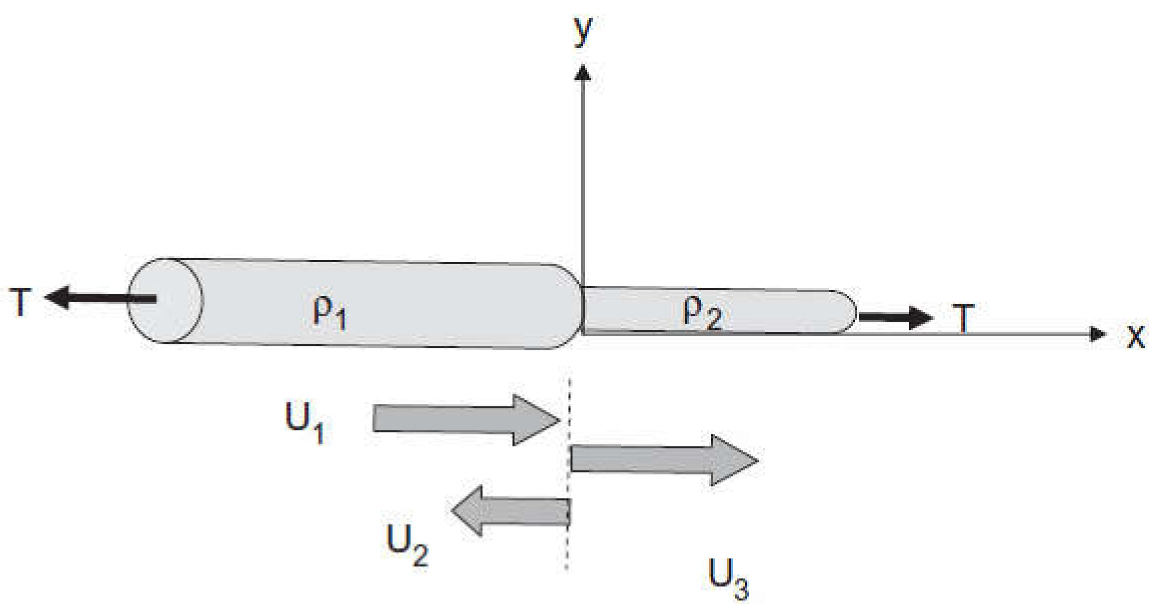
$$U_2 = A_2 \cdot e^{j\omega\left(t + \frac{x}{c_1}\right)}$$

$$U_3 = B \cdot e^{j\omega\left(t - \frac{x}{c_2}\right)}$$



Wave reflection (echo) in a 1D medium

- $y_1(x,t) = U_1 + U_2$ and $y_2(x,t) = U_3$
- Displacement continuity: $y_1|_{x=0} = y_2|_{x=0}$
- Equality of forces along y: $T \cdot \frac{\partial y_1}{\partial x}\Big|_{x=0} = T \cdot \frac{\partial y_2}{\partial x}\Big|_{x=0}$



Wave reflection (echo) in a 1D medium

- Boundary conditions:

$$A_1 \cdot e^{j\omega t} + A_2 \cdot e^{j\omega t} = B \cdot e^{j\omega t} \Rightarrow A_1 + A_2 = B$$

$$-\frac{j\omega}{c_1} A_1 e^{j\omega t} + \frac{j\omega}{c_1} A_2 e^{j\omega t} = -\frac{j\omega}{c_2} B e^{j\omega t} \Rightarrow \frac{(A_1 - A_2)}{c_1} = \frac{B}{c_2}$$

- Thus: $\frac{A_2}{A_1} = \frac{c_2 - c_1}{c_2 + c_1}$ and $c_1 = \sqrt{\frac{T}{\rho_1}}$, $c_2 = \sqrt{\frac{T}{\rho_2}}$

- Thus final reflection coeff.: $\frac{A_2}{A_1} = \frac{C_2 - C_1}{C_2 + C_1} = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}}$

Wave reflection (echo) in a 1D medium

- Similarly the transmission coefficient could be obtained:

$$\frac{B}{A_1} = \frac{2c_2}{c_1 + c_2}$$

- or

$$\frac{B}{A_1} = \frac{2\sqrt{\rho_1}}{\sqrt{\rho_1} + \sqrt{\rho_2}}$$

Wave reflection (echo) in a 1D medium

- Special case #1: a point connecting 2 similar strings ($\rho_1 = \rho_2 = \rho$.)

$$\frac{A_2}{A_1} = \frac{\sqrt{\rho} - \sqrt{\rho}}{\sqrt{\rho} + \sqrt{\rho}} = 0$$

There is no echo!

$$\frac{B}{A_1} = \frac{2\sqrt{\rho}}{\sqrt{\rho} + \sqrt{\rho}} = 1$$

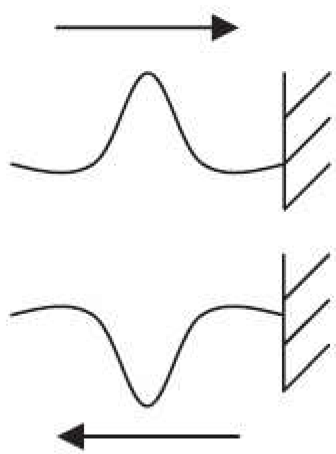
The transmitted wave is equal to the incident wave!

Wave reflection (echo) in a 1D medium

- Special case #2: a point is connected to a wall (a fixed end, $\rho_2 \to \infty$)

$$\frac{A_2}{A_1} = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}} = -1$$

$$\frac{B}{A_1} = \frac{2\sqrt{\rho_1}}{\infty} = 0$$

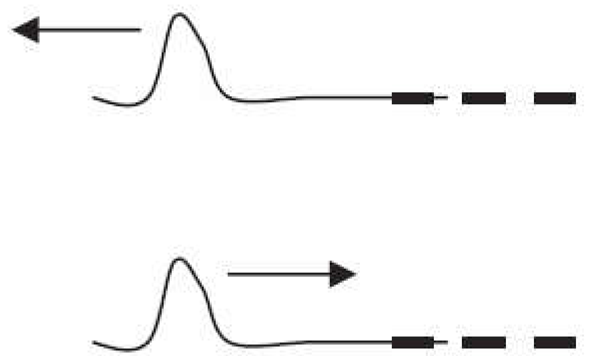


Wave reflection (echo) in a 1D medium

- Special case #3: The string has free end ($\rho_2 = 0$)

$$\frac{A_2}{A_1} = \frac{\sqrt{\rho_1} - \sqrt{\rho_2}}{\sqrt{\rho_1} + \sqrt{\rho_2}} = +1$$

(Naturally there is no transmitted wave)



Wave reflection (echo) in a 1D medium

- Special case #4: The second string is denser than the first one ($\rho_1 < \rho_2$):

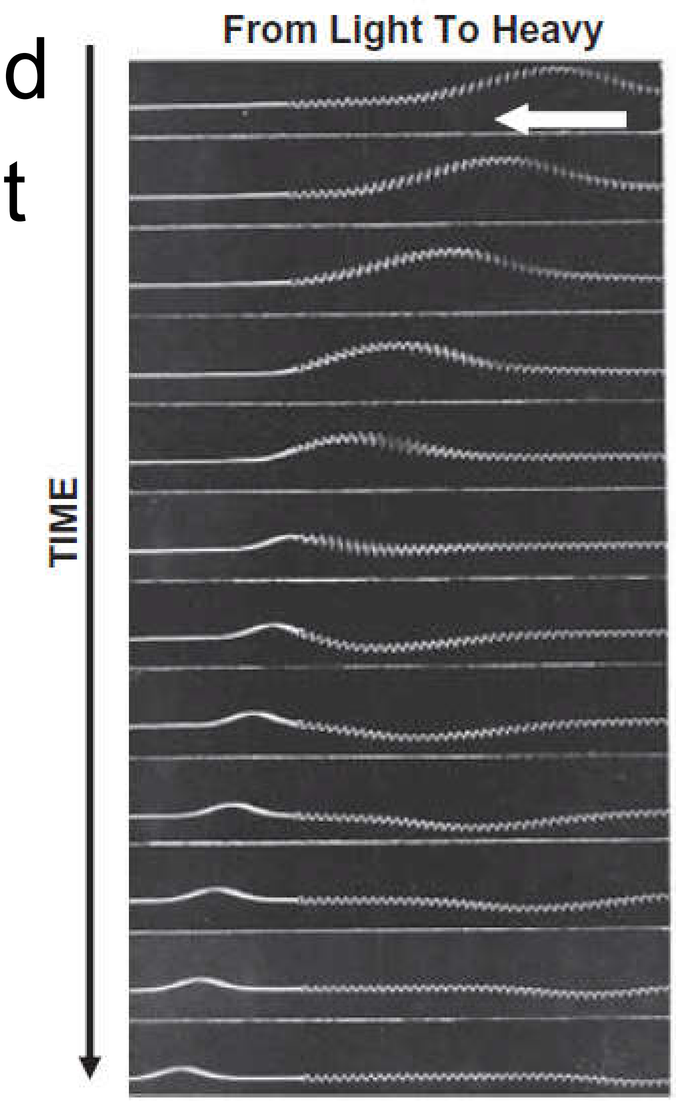
$A_2/A_1 < 0$

- The transmitted wave is smaller than the impinging wave:

$B < A_1$

Wave reflection (echo) in a 1D medium

- Special case #4: The second
- string is denser than the first
- one ($\rho_1 < \rho_2$):



Wave reflection (echo) in a 1D medium

- Special case #5: The first string is denser than the second one ($\rho_1 > \rho_2$):

$A_2/A_1 > 0$

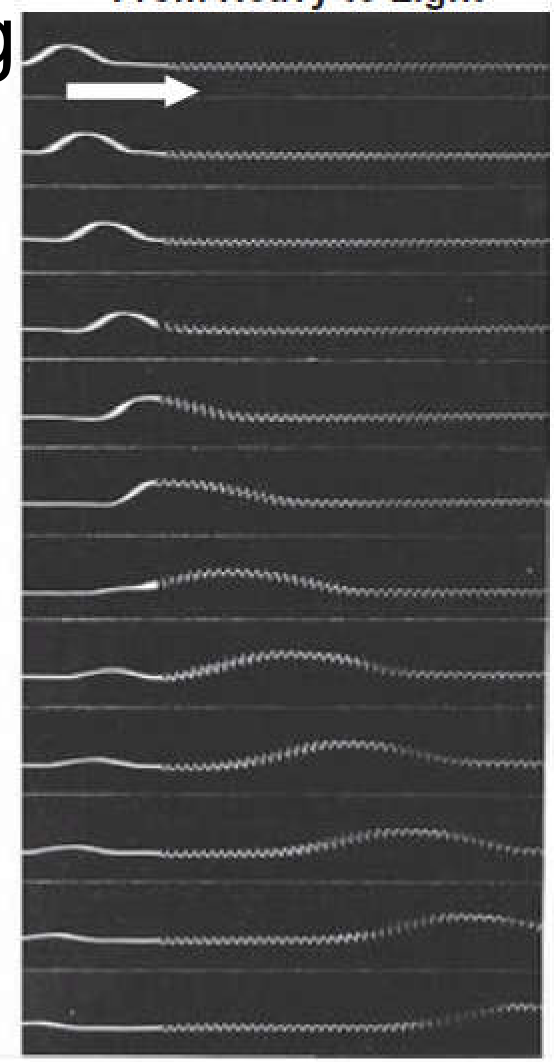
- The transmitted wave will be bigger than the impinging wave:

$B > A_1$

Wave reflection (echo) in a 1D medium

- Special case #5: The first string
- is denser than the second one

  ($\rho_1 > \rho_2$):



Wave energy in string

- Kinetic energy (related to velocity) + Potential energy (stemming from the string tension)
- For the impinging wave:

$$\frac{dU_1}{dt} \equiv \dot{U}_1 = j\omega \cdot U_1$$

$$\dot{U}_{1\max}^2 = \dot{U}_1 \cdot \dot{U}_1^* = \omega^2 \cdot A_1^2$$

$$E_I = \frac{1}{2}\rho_1 \cdot A_1^2 \cdot \omega^2$$

$$E_R = \frac{1}{2}\rho_1 \cdot A_2^2 \cdot \omega^2$$

$$E_T = \frac{1}{2}\rho_2 \cdot B^2 \cdot \omega^2$$

Wave energy in string

- Energy flowing into the point of discontinuity in time equals the energy flowing out of it:

$$E_I \cdot C_1 = E_R \cdot C_1 + E_T \cdot C_2$$

- Or

$$\frac{1}{2}\rho_1 c_1 \cdot \left(A_1^2 \omega^2\right) = \frac{1}{2}\rho_1 c_1 \cdot \left(A_2^2 \omega^2\right) + \frac{1}{2}\rho_2 c_2 \cdot \left(B^2 \omega^2\right)$$

- Which yields ($Z \triangleq \rho \cdot C$):

$$Z_1 A_1^2 = Z_1 A_2^2 + Z_2 B^2$$

Wave energy in string

- Recalling the refl. And trans. ratios:

$$\frac{A_2}{A_1} = \frac{c_2 - c_1}{c_2 + c_1}$$

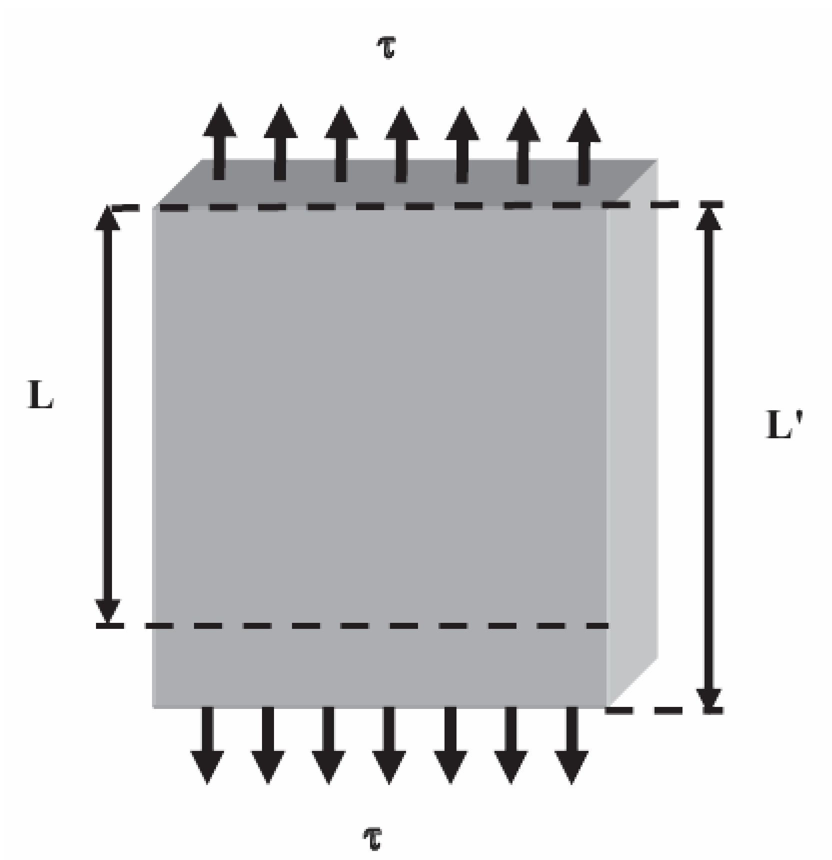
$$\frac{B}{A_1} = \frac{2c_2}{c_1 + c_2}$$

- Which yields:

$$\frac{\text{Reflected energy}}{\text{Impinging energy}} = \frac{(Z_1 - Z_2)^2}{(Z_1 + Z_2)^2}$$

$$\frac{\text{Through-transmitted energy}}{\text{Impinging energy}} = \frac{4Z_1Z_2}{(Z_1 + Z_2)^2}$$

Longitudinal wave propagation



Longitudinal wave propagation

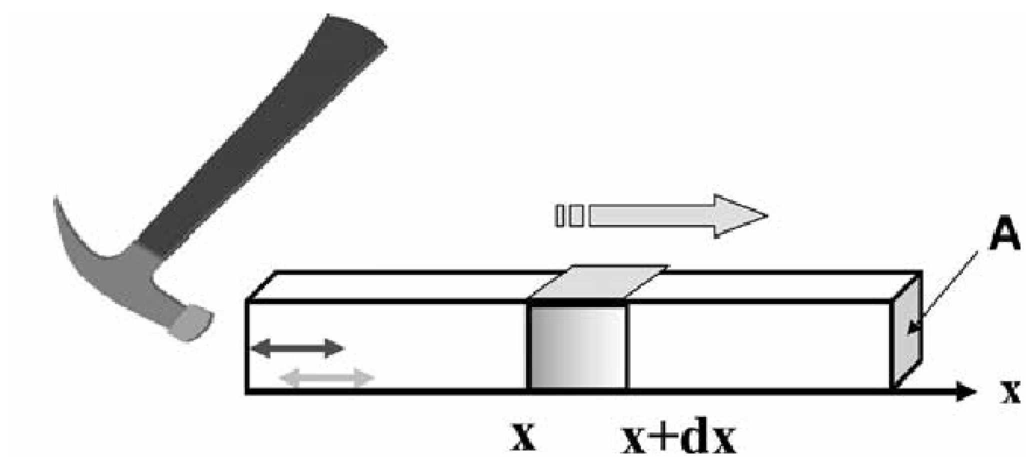
- The strain ε :

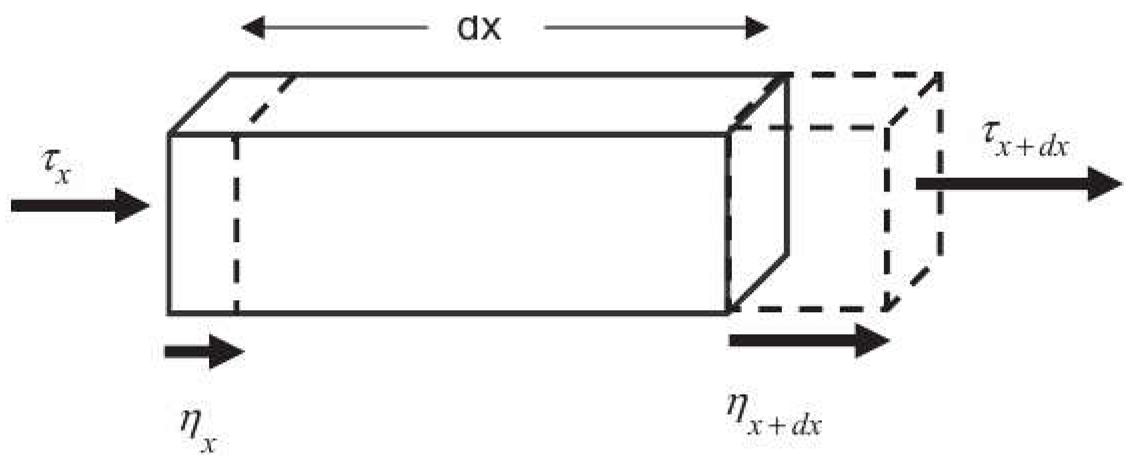
$$\varepsilon = \frac{L' - L}{L} = \frac{\Delta L}{L}$$

- Hook's law (E: Young's modulus or modulus of elasticity)

$$\varepsilon = \frac{\tau}{E}$$

Longitudinal wave propagation





Longitudinal wave propagation

- $\tau = E \cdot \varepsilon$

$$F_{\text{left}} = A\tau_x = A \cdot E \cdot \varepsilon = A \cdot E \cdot \left.\frac{\partial \eta}{\partial x}\right|_x$$

$$F_{\text{right}} = A \cdot E \cdot \left.\frac{\partial \eta}{\partial x}\right|_{x+dx}$$

$$\Delta F = F_{\text{right}} - F_{\text{left}} = \frac{\partial^2 \eta}{\partial t^2}$$

Longitudinal wave propagation

- $$\Rightarrow AE \cdot \left(\left.\frac{\partial \eta}{\partial X}\right|_{x+dx} - \left.\frac{\partial \eta}{\partial X}\right|_{x} \right) = \underbrace{\rho \cdot A \cdot dx}_{m} \cdot \ddot{\eta} \;\Big|\; \cdot \frac{1}{A\,dx}$$
  $$\Rightarrow E \cdot \left(\frac{\left.\frac{\partial \eta}{\partial X}\right|_{x+dx} - \left.\frac{\partial \eta}{\partial X}\right|_{x}}{dx} \right) = \rho \cdot \ddot{\eta}$$
- $$\boxed{E \cdot \frac{\partial^2 \eta}{\partial x^2} = \rho \cdot \ddot{\eta}}$$: Wave equation for the rod

Longitudinal wave propagation

- Wave equation for the rode:

$$\frac{\partial^2 \eta}{\partial x^2} = \frac{1}{\frac{E}{\rho}} \cdot \ddot{\eta} \Rightarrow \frac{1}{C^2} = \frac{\rho}{E}$$

and $$\boxed{C = \sqrt{\frac{E}{\rho}}}$$